

Approximate formulas for total cross section for moderately small eikonal

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Abstract

The eikonal representation for the total cross section is considered. The approximate formulas for a moderately small eikonal are derived. In contrast to standard eikonal integrals, they contain no Bessel functions, and, hence, no rapidly oscillating integrands. The formulas obtained are applied to numerical evaluations of the total cross section for a number of particular expressions for the eikonal. It is shown that for pure imaginary eikonals the relative error of $O(10^{-5})$ can be achieved. Also two improper triple integrals are analytically calculated.

1 Introduction

In potential scattering the eikonal approximation is applied when a scattering angle is small, and energy of incoming particle is much larger than a “strength” of an interaction potential [1], [2]. In quantum field theory the use of the eikonal approximation is justified if

$$-t/s \ll 1, \quad (1)$$

where s, t are Mandelstam variables. In perturbation theory the eikonal representation was studied in [3]. It can be derived in the framework of quasipotential approach for small scattering angles and smooth quasipotentials [4].

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In the Regge approach [5], the eikonalization corresponds to the summation of contributions of multi-reggeon exchanges. The off-shell extension of the Regge-eikonal approach is given in [6].

In the eikonal approximation, the total cross section is given by the formula

$$\sigma_{\text{tot}}(s) = 4\pi \int_0^\infty b db \{1 - \exp[-\text{Im}\chi(s, b)] \cos[\text{Re}\chi(s, b)]\} , \quad (2)$$

where b is an impact parameter. The eikonal $\chi(s, b)$ is related to the Born amplitude $A_B(s, t)$ by the Fourier-Bessel transformation,

$$\chi(s, b) = \frac{1}{4\pi s} \int_0^\infty dq_\perp q_\perp J_0(q_\perp b) A_B(s, q_\perp) . \quad (3)$$

where we defined $q_\perp = \sqrt{-t}$. In its turn, the Born amplitude is defined via eikonal as

$$A_B(s, q_\perp) = 4\pi s \int_0^\infty b db J_0(q_\perp b) \chi(s, b) . \quad (4)$$

As we see from eqs. (2)-(4), in order to calculate the total cross section, one has to deal with integrals containing *rapidly oscillating* Bessel functions. The difficulties in evaluating integrals over half-line involving products of Bessel functions were considered in refs. [7], [8]. Note that integrals with the products of the Bessel functions are very difficult to evaluate numerically because of their poor convergence and oscillatory nature.

As it was recently shown in [9], this problem can be solved for moderately small eikonal. We call it *moderately small* eikonal, if it is less than unity, $|\chi(s, b)| < 1$, but the inequality $|\chi(s, b)| \ll 1$ is not implied.¹ In ref. [9] approximate formulas for scattering amplitudes were derived which contain no Bessel functions, and, hence, no rapidly oscillating integrands. In the present paper we will derive analogous approximate formulas for the total cross section (2) which can be used for numerical calculations of $\sigma_{\text{tot}}(s)$.

The paper is organized as follows. In Section 2 the approximate formulas for the total cross section are derived. In Section 3 these formulas are used for numerical calculations of σ_{tot} for a variety of eikonal functions χ in order to

¹The *small* eikonal, $|\chi(s, b)| \ll 1$, is not so interesting, since corresponding formulas are considerably simplified for this case.

study their efficiency. Several relevant formulas are collected in Appendix A. In Appendix B two improper triple integrals are calculated.

2 Approximate formula for the cross section

In what follows we will assume that the eikonal $\chi(s, b)$ is moderately small.² Let us start from the expansion of the integrand in the r.h.s. of eq. (2):

$$\begin{aligned}
1 - e^{-\text{Im}\chi} \cos(\text{Re}\chi) &= \text{Im}\chi + \frac{1}{2} [\text{Re}\chi^2 - \text{Im}\chi^2] - \frac{1}{6} \text{Im}\chi [3\text{Re}\chi^2 - \text{Im}\chi^2] \\
&+ \frac{1}{24} [6 \text{Re}\chi^2 \text{Im}\chi^2 - \text{Re}\chi^4 - \text{Im}\chi^4] \\
&- \frac{1}{120} \text{Im}\chi [10 \text{Re}\chi^2 \text{Im}\chi^2 - 5 \text{Re}\chi^4 - \text{Im}\chi^4] \\
&- \frac{1}{720} [15 \text{Re}\chi^4 \text{Im}\chi^2 - 15 \text{Re}\chi^2 \text{Im}\chi^4 - \text{Re}\chi^6 + \text{Im}\chi^6] + \dots
\end{aligned} \tag{5}$$

We omitted higher terms in (5). As we will see in the next section, an account of only four terms in this expansion is enough to approximate σ_{tot} with a small relative error.

Correspondingly, we obtain the following expansion for the cross section:

$$\sigma_{\text{tot}}(s) = \sigma_1(s) + \dots + \sigma_6(s) + \dots \tag{6}$$

Let us define the integral with the product of several Bessel functions:

$$F_n(a_1, \dots, a_n) = \int_0^\infty dx x \prod_{k=1}^n J_0(a_k x), \tag{7}$$

where $a_k > 0$, $k = 1, \dots, n$. Explicit expressions for $F_n(a_1, \dots, a_n)$, $n = 2, \dots, 6$, are presented in Appendix A. Then we find from eqs. (5), (2), (3) and (6):

$$\sigma_1(s) = \frac{\text{Im}A_B(s, 0)}{s}, \tag{8}$$

$$\sigma_2(s) = \frac{1}{2^3 \pi s^2} \int_0^\infty dq q \{ [\text{Re}A_B(s, q)]^2 - [\text{Im}A_B(s, q)]^2 \}. \tag{9}$$

²The definition of the moderate small eikonal is given in Introduction.

In deriving (9), we used eq. (A.1) from Appendix A. The other four terms in (6) are given by formulas:

$$\begin{aligned} \sigma_3(s) = \frac{1}{2^5 \pi^2 s^3} \prod_{i=1}^3 \int_0^\infty dq_i q_i F_3(q_1, q_2, q_3) & \left[\frac{1}{3} \prod_{i=1}^3 \text{Im} A_B(s, q_i) \right. \\ & \left. - \text{Im} A_B(s, q_1) \prod_{i=2}^3 \text{Re} A_B(s, q_i) \right], \end{aligned} \quad (10)$$

$$\begin{aligned} \sigma_4(s) = \frac{1}{2^8 \pi^3 s^4} \prod_{i=1}^4 \int_0^\infty dq_i q_i F_4(q_1, q_2, q_3, q_4) & \left[\prod_{i=1}^2 \text{Im} A_B(s, q_i) \prod_{i=3}^4 \text{Re} A_B(s, q_i) \right. \\ & \left. - \frac{1}{6} \prod_{i=1}^4 \text{Re} A_B(s, q_i) - \frac{1}{6} \prod_{i=1}^4 \text{Im} A_B(s, q_i) \right], \end{aligned} \quad (11)$$

$$\begin{aligned} \sigma_5(s) = \frac{1}{3 \cdot 2^{10} \pi^4 s^5} \prod_{i=1}^5 \int_0^\infty dq_i q_i F_5(q_1, q_2, q_3, q_4, q_5) & \left[\frac{1}{10} \prod_{i=1}^5 \text{Im} A_B(s, q_i) \right. \\ & + \frac{1}{2} \text{Im} A_B(s, q_1) \prod_{i=2}^5 \text{Re} A_B(s, q_i) \\ & \left. - \prod_{i=1}^3 \text{Im} A_B(s, q_i) \prod_{i=4}^5 \text{Re} A_B(s, q_i) \right], \end{aligned} \quad (12)$$

$$\begin{aligned} \sigma_6(s) = \frac{1}{3 \cdot 2^{14} \pi^5 s^6} \prod_{i=1}^6 \int_0^\infty dq_i q_i F_6(q_1, q_2, q_3, q_4, q_5, q_6) & \left[-\frac{1}{15} \prod_{i=1}^6 \text{Im} A_B(s, q_i) \right. \\ & + \frac{1}{15} \prod_{i=1}^6 \text{Re} A_B(s, q_i) - \prod_{i=1}^2 \text{Im} A_B(s, q_i) \prod_{i=3}^6 \text{Re} A_B(s, q_i) \\ & \left. + \prod_{i=1}^4 \text{Im} A_B(s, q_i) \prod_{i=5}^6 \text{Re} A_B(s, q_i) \right]. \end{aligned} \quad (13)$$

The explicit expressions for the functions F_i ($i = 3, 4, 5, 6$) are given by eqs. (A.3), (A.5), (A.9) and (A.10).

3 Numerical evaluation of the cross section

In this section we will numerically calculate the total cross section for a variety of expressions taken for the eikonal. The main goal is to verify whether our formulas approximate σ_{tot} well or not.

1.1. We start from the case when the eikonal is a function of b only. As the first example, let us take $\text{Re}\chi(b) = 0$,

$$\text{Im}\chi(b) = \frac{b_0^4}{(b_0^2 + b^2)^2} . \quad (14)$$

Thus, $\chi(0) = 1$, and $|\chi(b)| < 1$ for $b > 0$. It what follows, it will be assumed that the impact parameter b is measured in units of b_0 . Then we can write³

$$\text{Re}\chi(b) = 0 , \quad \text{Im}\chi(b) = \frac{1}{(1 + b^2)^2} . \quad (15)$$

The eikonal (15) corresponds to the following Born amplitude⁴

$$\text{Re}A_B(q) = 0 , \quad \text{Im}A_B(q) = 4\pi s \int_0^\infty b db J_0(qb) \frac{1}{(1 + b^2)^2} = 2\pi s q K_1(q) , \quad (16)$$

where $K_1(z)$ is the modified Bessel function of the second kind (Macdonald function).

Correspondingly, the total cross section is equal to

$$\begin{aligned} \sigma_{\text{tot}} &= 4\pi s \int_0^\infty b db \left\{ 1 - \exp \left[-\frac{1}{(1 + b^2)^2} \right] \right\} \\ &= \pi \int_0^1 dz z^{-3/2} (1 - e^{-z}) = 2\pi [2 \text{Erf}(1) + e^{-1} - 1] = 5.413138 . \end{aligned} \quad (17)$$

Here $\text{Erf}(z) = \int_0^z e^{-t^2} dt$ is the Gauss error function[10].

Consider several first terms in expansion (6). Since $\lim_{z \rightarrow 0} [z K_1(z)] = 1$ [12], we immediately get

$$\sigma_1 = \frac{\text{Im}A_B(0)}{s} = 2\pi . \quad (18)$$

³Correspondingly, the momentum q will be measured in units of b_0^{-1} .

⁴In initial units, $\text{Im}A_B(q) = 2\pi s q b_0^3 K_1(q b_0)$.

We also find that

$$\sigma_2 = -\frac{\pi}{2} \int_0^\infty [xK_1(x)]^2 x dx = -\frac{\pi}{3}, \quad (19)$$

and

$$\begin{aligned} \sigma_3 = & \frac{1}{3} \int_0^\infty dx x^2 K_1(x) \int_0^\infty dy y^2 K_1(y) \int_0^\infty dz z^2 K_1(z) \\ & \times \frac{\theta([z^2 - (x-y)^2][(x+y)^2 - z^2])}{\sqrt{[z^2 - (x-y)^2][(x+y)^2 - z^2]}} = \frac{1}{3} \times \frac{\pi}{5} = \frac{\pi}{15}. \end{aligned} \quad (20)$$

The details of the calculation of this integral are presented in Appendix B.

Let us define the relative error

$$\varepsilon = \frac{|\sigma - \sum_{i=1}^k \sigma_i|}{\sigma}, \quad (21)$$

(in what follows, $k = 3, 4$). Then the sum

$$\sigma_1 + \sigma_2 + \sigma_3 = \frac{26\pi}{15} = 5.445427 \quad (22)$$

results in $\varepsilon = 6.0 \cdot 10^{-3}$. The next term in (6) is equal to⁵

$$\begin{aligned} \sigma_4 = & -\frac{1}{96\pi} \int_0^\infty dx x^2 K_1(x) \int_0^\infty dy y^2 K_1(y) \int_0^\infty dz z^2 K_1(z) \int_0^\infty du u^2 K_1(u) \\ & \times [\pi^2 F_4(x, y, z, u)] = -\frac{1}{96\pi} \times 12.512106 = -0.041486, \end{aligned} \quad (23)$$

where the function $F_4(x, y, z, u)$ is defined by eq. (A.5). With the account of σ_4 , the relative error becomes several times smaller, $\varepsilon = 1.7 \cdot 10^{-3}$.

1.2. For the pure real eikonal,

$$\text{Re}\chi(b) = \frac{1}{(1+b^2)^2}, \quad \text{Im}\chi(b) = 0, \quad (24)$$

the cross section is

$$\sigma_{\text{tot}} = \pi \int_0^1 dz z^{-3/2} (1 - \cos z) = 2\pi \left[\sqrt{2\pi} S(1) + \cos(1) - 1 \right] = 1.010581. \quad (25)$$

⁵Here and below four-dimensional integrals are numerically calculated with the help of the program Mathematica.

Here $S(z) = (1/\sqrt{2\pi}) \int_0^z t^{-1/2} \sin t dt$ is the Fresnel integral [10]. Since $\text{Im}\chi = 0$, we get $\sigma_1 = \sigma_3 = 0$, while

$$\sigma_2 = \frac{\pi}{3} = 1.047198 . \quad (26)$$

As one can see from, the leading term (26) estimates the cross section with the precision $\varepsilon = 3.5 \cdot 10^{-2}$. The next non-zero term,

$$\sigma_4 = -0.041489 , \quad (27)$$

reduces the relative error to the value of $\varepsilon = 4.6 \cdot 10^{-3}$.

1.3. Now let us put

$$\text{Re}\chi(b) = \text{Im}\chi(b) = \frac{1}{(1+b^2)^2} . \quad (28)$$

Then

$$\begin{aligned} \sigma_{\text{tot}} &= \pi \int_0^1 dz z^{-3/2} (1 - e^{-z} \cos z) = 2\pi \{-1 + e^{-1} \cos(1) \\ &\quad + i[\sqrt{(1+i)} \text{Erfi}(\sqrt{-1-i}) - \sqrt{(1-i)} \text{Erfi}(\sqrt{-1+i})]\} \\ &= 5.991252 , \end{aligned} \quad (29)$$

where $\text{Erfi}(z) = -i \text{Erf}(iz)$. We find that

$$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = \frac{28\pi}{15} + 0.165956 = 6.030262 , \quad (30)$$

with $\varepsilon = 6.5 \cdot 10^{-3}$.

Thus, for the case $|\chi(b)| \leq 1$ the account of four terms in expansion (6) approximates the total cross section with the relative errors of $O(10^{-3})$. Here and in what follow, the notation $a = O(10^{-n})$ means that a is a term of the order $(-n)$, i.e. $10^{-n} \leq |a| < 10^{-n+1}$.⁶

2.1. Let us now consider the case when $|\chi(b)| < 1/4$ for all $b \geq 0$, and take as example the eikonal:

$$\text{Re}\chi(b) = 0 , \quad \text{Im}\chi(b) = \frac{1}{(2+b^2)^2} . \quad (31)$$

⁶Don't confuse with Landau's symbol used to describe the asymptotic behavior of functions [11].

The exact value of the cross section is

$$\begin{aligned}\sigma_{\text{tot}} &= \pi \int_0^{1/4} dz z^{-3/2} (1 - e^{-z}) = 4\pi \left[\text{Erf} \left(\frac{1}{2} \right) + e^{-1/4} - 1 \right] \\ &= 3.016957 .\end{aligned}\tag{32}$$

On the other hand,

$$\sigma_1 + \sigma_2 + \sigma_3 = \frac{461\pi}{480} = 3.017238 ,\tag{33}$$

with the relative error $\varepsilon = 9.3 \cdot 10^{-5}$. For four terms, we get

$$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = \frac{461\pi}{480} - 0.000324 = 3.016914 ,\tag{34}$$

with $\varepsilon = 1.4 \cdot 10^{-5}$.

2.2. For a pure real eikonal, satisfying $|\chi| < 1/4$,

$$\text{Re}\chi(b) = \frac{1}{(2+b^2)^2} , \quad \text{Im}\chi(b) = 0 ,\tag{35}$$

we find:

$$\begin{aligned}\sigma_{\text{tot}} &= \pi \int_0^{1/4} dz z^{-3/2} (1 - \cos z) = 2\pi \left[-2 + 2 \cos \left(\frac{1}{4} \right) + \sqrt{2\pi} \text{S} \left(\frac{1}{4} \right) \right] \\ &= 0.130608 .\end{aligned}\tag{36}$$

In such a case, $\sigma_1 = \sigma_3 = 0$,

$$\sigma_2 + \sigma_4 = \frac{\pi}{24} - 0.000324 = 0.130576 ,\tag{37}$$

and $\varepsilon = 2.5 \cdot 10^{-4}$.

2.3. Now consider the case when the imaginary and real parts of the eikonal are equal:

$$\text{Re}\chi(b) = \text{Im}\chi(b) = \frac{1}{(2+b^2)^2} .\tag{38}$$

Then

$$\begin{aligned}\sigma_{\text{tot}} &= \pi \int_0^{1/4} dz z^{-3/2} (1 - e^{-z} \cos z) = 4\pi [-1 + e^{-1/4} \cos(1/4)] \\ &\quad + 2^{1/4} \pi^{3/2} \{ (-1)^{1/8} \text{Erf} [(-1)^{1/8} 2^{-3/4}] + (-1)^{-1/8} \text{Erf} [(-1)^{-1/8} 2^{-3/4}] \} \\ &= 3.129626 .\end{aligned}\tag{39}$$

We find that $\sigma_2 = 0$, and

$$\sigma_1 + \sigma_3 + \sigma_4 = \frac{239\pi}{240} + 0.000648 = 3.129151 ,\tag{40}$$

with $\varepsilon = 1.5 \cdot 10^{-4}$.

We can conclude that for moderately small eikonal ($|\chi(b)| \leq 1/4$, in our case), the account of four first terms in expansion (6) results in small relative error of $O(10^{-4})$, or even $O(10^{-5})$ (for pure imaginary eikonal).

3.1. Up to now we considered only b -dependence of the eikonal. In general, the eikonal depends both on the impact parameter b and invariant energy s . Let the eikonal to depend on s via dimensionless parameter $a = a(s)$. For simplicity, we assume that the eikonal has a factorized representation. We start from pure imaginary eikonal:

$$\text{Re}\chi(a, b) = 0 , \quad \text{Im}\chi(a, b) = \frac{a}{(1 + b^2)^{3/2}} .\tag{41}$$

As an example, we can take⁷

$$a = \frac{1}{1 + (\pi/20) \ln(s/s_0)} ,\tag{42}$$

where $s \geq s_0 = 1\text{GeV}^2$. Then the values of $a = 1, 0.5, 0.3$, and 0.25 will correspond to energies $\sqrt{s} = 1\text{GeV}, 24\text{ GeV}, 2\text{ TeV}$ and 14 TeV , respectively.

We obtain

$$\begin{aligned}\sigma_{\text{tot}}(a) &= \frac{4\pi}{3} a^{2/3} \int_0^a dz z^{-5/3} (1 - e^{-z}) \\ &= 2\pi \left[-1 + e^{-a} + a^{2/3} \Gamma\left(\frac{1}{3}\right) - a^{2/3} \Gamma\left(\frac{1}{3}, a\right) \right] ,\end{aligned}\tag{43}$$

⁷In particular, it means that $A_B \sim s/\ln s$ at large s .

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ is the incomplete gamma function [10]. We also find that

$$\text{Im}A_B(a, q) = 4\pi s a \int_0^\infty b db J_0(qb) \frac{1}{(1+b^2)^{3/2}} = 4\pi s a e^{-q}. \quad (44)$$

In particular, we get

$$\sigma_{\text{tot}}(1) = 11.249497. \quad (45)$$

Correspondingly, the corrections $\sigma_i(a=1)$ in expansion (6) look like

$$\sigma_1(1) = 4\pi, \quad (46)$$

$$\sigma_2(1) = -2\pi \int_0^\infty e^{-2x} x dx = -\frac{\pi}{2}, \quad (47)$$

and

$$\begin{aligned} \sigma_3(1) &= \frac{4}{3} \int_0^\infty dx x e^{-x} \int_0^\infty dy y e^{-y} \int_0^\infty dz z e^{-z} \\ &\times \frac{\theta([z^2 - (x-y)^2][(x+y)^2 - z^2])}{\sqrt{[z^2 - (x-y)^2][(x+y)^2 - z^2]}} = \frac{4}{3} \times \frac{\pi}{14} = \frac{2\pi}{21}. \end{aligned} \quad (48)$$

The improper triple integral in (48) is calculated in Appendix B, see eqs. (B.12)-(B.14). As a result, we find that

$$\sigma_1(1) + \sigma_2(1) + \sigma_3(1) = \frac{151\pi}{42} = 11.294774, \quad (49)$$

with $\varepsilon = 4.0 \cdot 10^{-3}$. The account of σ_4 ,

$$\begin{aligned} \sigma_4(1) &= -\frac{1}{6\pi} \int_0^\infty dx x e^{-x} \int_0^\infty dy y e^{-xy} \int_0^\infty dz z e^{-z} \int_0^\infty du u e^{-u} \\ &\times [\pi^2 F_4(x, y, z, u)] = -\frac{1}{6\pi} \times 1.422443 = -0.075463, \end{aligned} \quad (50)$$

reduces the relative error slightly to the value of $\varepsilon = 2.7 \cdot 10^{-3}$.

3.2. Consider another “energy” and take $a = 1/3$. We get from eq. (43) that $\sigma_{\text{tot}}(1/3) = 4.024724$. As for the approximate formulas, we find from them that

$$\sigma_1(1/3) + \sigma_2(1/3) + \sigma_3(1/3) = \frac{1453\pi}{1134} = 4.025339, \quad (51)$$

with $\varepsilon = 1.5 \cdot 10^{-4}$, and

$$\begin{aligned} & \sigma_1(1/3) + \sigma_2(1/3) + \sigma_3(1/3) + \sigma_4(1/3) \\ &= \frac{1453\pi}{1134} - 9.32 \cdot 10^{-4} = 4.024407 , \end{aligned} \quad (52)$$

with $\varepsilon = 7.9 \cdot 10^{-5}$.

3.3 Let us put $a = 1/4$ in eq. (41).⁸ In such a case, eq. (43) results in $\sigma_{\text{tot}}(1/4) = 3.047896$. Correspondingly,

$$\sigma_1(1/4) + \sigma_2(1/4) + \sigma_3(1/4) = \frac{163\pi}{168} . \quad (53)$$

Thus, we obtain rather small relative error of $6.5 \cdot 10^{-5}$. After adding one more term, we get

$$\begin{aligned} & \sigma_1(1/4) + \sigma_2(1/4) + \sigma_3(1/4) + \sigma_4(1/4) \\ &= \frac{163\pi}{168} - 2.95 \cdot 10^{-4} = 3.047795 . \end{aligned} \quad (54)$$

Thus, the account of σ_4 cuts the relative error in half to the value of $3.3 \cdot 10^{-5}$.

4.1. Now let us consider the pure real eikonal of the form ($a = 1/4$):

$$\text{Re}\chi(b) = \frac{1}{4(1+b^2)^{3/2}} , \quad \text{Im}\chi(b) = 0 . \quad (55)$$

The total cross section is now equal to

$$\begin{aligned} \sigma_{\text{tot}} &= \frac{\pi}{4} [iE_{2/3}(-i/4) - iE_{2/3}(i/4) + 2^{2/3}\Gamma(1/3) + 8\cos(1/4) - 8] \\ &= 0.097971 , \end{aligned} \quad (56)$$

where $E_n(z) = \int_1^\infty e^{-tz}t^{-n}dt = z^{n-1}\Gamma(1-n, z)$ is the exponential integral function [12]. As for the terms σ_i , we obtain $\sigma_1 = \sigma_3 = 0$, and

$$\sigma_2 = \frac{\pi}{32} , \quad \sigma_4 = -2.95 \cdot 10^{-4} , \quad (57)$$

that results in $\varepsilon = 9.3 \cdot 10^{-4}$.

⁸Remember that for parametrization (42), $a = 1/4$ corresponds to $s = 14$ TeV.

4.2. Finally, let us consider the eikonal which has both the real and imaginary part ($a = 1/4$):

$$\text{Re}\chi(b) = \text{Im}\chi(b) = \frac{1}{4(1+b^2)^{3/2}}, \quad (58)$$

with

$$\begin{aligned} \sigma_{\text{tot}} &= \frac{\pi}{3} \left\{ 2E_{5/3}[(1-i)/4] + 2E_{5/3}[(1+i)/4] - \sqrt{3}\Gamma(-2/3) - 6 \right\} \\ &= 3.133029. \end{aligned} \quad (59)$$

In such a case, $\sigma_2 = 0$, and

$$\sigma_1 = \pi, \quad \sigma_3 = -\frac{\pi}{336}, \quad \sigma_4 = 1.18 \cdot 10^{-3}, \quad (60)$$

As a result, we obtain the relative error of $\varepsilon = 1.3 \cdot 10^{-4}$.

4 Conclusions

In the present paper we studied the eikonal approximation for the total cross section. We derived the approximate formulas for the case when the eikonal χ is moderately small (see eqs. (6) and (9)-(13)). Their advantage with respect to the standard eikonal formulas is that our integrals contain no rapidly oscillating Bessel functions. To demonstrate the efficiency of these approximate formulas, we applied them for the calculations of the total cross sections for a number of particular eikonal parametrizations.

It is just not possible to investigate all possible expressions for the eikonal that may arise in real calculations of the cross sections. Nevertheless, the results obtained in Section 3 allow us to make some conclusions. If the eikonal is indeed moderately small (say, $|\chi| \leq 1/4$ for all b), then the sum $\sum_{i=1}^4 \sigma_i$ in (6) approximates the total cross section with the relative error of $O(10^{-5})$ for the pure imaginary eikonals.⁹ For the pure real eikonals, the precision is noticeably less. For the case $\text{Re}\chi = \text{Im}\chi$, the relative error of $O(10^{-4})$ can be achieved by using four first terms in the expansion (6). In order to improve the precision, one has to take into account additional terms (12), (13) in (6). For *not* a small eikonal ($|\chi| \leq 1$, with $|\chi(b)| \simeq 1$ for $b \simeq 0$), the relative error is at least by one order in magnitude larger.¹⁰

⁹The notation $O(10^{-n})$ is explained in Section 3 after eq. (30).

¹⁰It is a bit surprising that in this case the sum $\sum_{i=1}^4 \sigma_i$ approximates σ_{tot} with the rather small relative error of $O(10^{-3})$.

Appendix A

The integral with the product of two Bessel functions is well-known (see eq. (3.108) in [13] or eq. 6.512.8. in [14])¹¹

$$F_2(a, b) = \int_0^\infty dx x J_0(ax) J_0(bx) = \frac{1}{a} \delta(a - b) . \quad (\text{A.1})$$

The integral with three Bessel functions $J_0(z)$ ($a, b, c > 0$),

$$F_3(a, b, c) = \int_0^\infty dx x J_0(ax) J_0(bx) J_0(cx) , \quad (\text{A.2})$$

is given by the formula (see eq. 13.46(3) in [15], as well as eqs. 2.12.42.14., 2.12.42.15. in [16]):

$$F_3(a, b, c) = \begin{cases} \frac{1}{2\pi\Delta_3}, & \Delta_3^2 > 0 , \\ 0, & \Delta_3^2 < 0 , \end{cases} \quad (\text{A.3})$$

where

$$16\Delta_3^2 = [c^2 - (a - b)^2][(a + b)^2 - c^2] . \quad (\text{A.4})$$

Note that the integral in (A.3) is divergent if $\Delta_3^2 = 0$ (it takes place when one of the parameters is equal to the sum of the others, say, $c = a + b > 0$).

Recently [9], we derived the full expression¹² for $F_4(a, b, c, d)$:

$$F_4(a, b, c, d) = \begin{cases} \frac{1}{\pi^2\Delta_4} \text{K}\left(\frac{\sqrt{abcd}}{\Delta_4}\right), & \Delta_4^2 > abcd , \\ \frac{1}{\pi^2\sqrt{abcd}} \text{K}\left(\frac{\Delta_4}{\sqrt{abcd}}\right), & 0 < \Delta_4^2 < abcd , \\ \frac{1}{2\pi\sqrt{abcd}} & \Delta_4^2 = 0 , \\ 0, & \Delta_4^2 < 0 . \end{cases} \quad (\text{A.5})$$

¹¹In ref. [9] this formula was generalized for the product of two Bessel functions J_ν with $\text{Re } \nu > -1$, $|\arg ab| < \pi/2$.

¹²The corresponding formula for F_4 in [16] did not include an important case $\Delta_4^2 < 0$.

where $a, b, c, d > 0$,

$$16\Delta_4^2 = (a + b + c - d)(a + b + d - c)(a + c + d - b)(b + c + d - a) , \quad (\text{A.6})$$

and

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (\text{A.7})$$

is the complete elliptic integral of the first kind ($0 \leq k < 1$) [17]. The quantity $F_4(a, b, c, d)$ (A.5) is not defined if $\Delta_4^2 = abcd$, since $K(k)$ diverges at $k \rightarrow 1$ [17]:

$$K(k)|_{k \rightarrow 1} \simeq \ln(4/\sqrt{1-k^2}) . \quad (\text{A.8})$$

Nevertheless, since $F_4(a, b, c, d)$ will be used only as a part of the integrand, we can safely put $F_4(a, b, c, d) = 0$ for $\Delta_4^2 = abcd$.

Finally, the following expressions were obtained in [9] ($a, b, c, d, e, f > 0$):

$$F_5(a, b, c, d, e) = \int_0^\infty dt \, t \, F_3(a, b, t) F_4(c, d, e, t) , \quad (\text{A.9})$$

$$F_6(a, b, c, d, e, f) = \int_0^\infty dt \, t \, F_4(a, b, c, t) F_4(d, e, f, t) . \quad (\text{A.10})$$

where $F_3(a_1, a_2, a_3)$ and $F_4(a_1, a_2, a_3, a_4)$ are defined by formulas (A.3) and (A.5), respectively. By doing in the same way, one can express the integrals

$$F_n(a_1, \dots, a_n) = \int_0^\infty dx \, x \prod_{k=1}^n J_0(a_k x) , \quad (\text{A.11})$$

with $n > 6$ and $a_k > 0$, $k = 1, \dots, n$ as a $(n-3)$ -dimensional integral of algebraic functions. Let us note that $F_n(a_1, \dots, a_n) = 0$ for any n , if $a_n > a_1 + a_2 + \dots + a_{n-1}$.

Appendix B

In this Appendix we will calculate two improper triple integrals mentioned in the main text. Consider the first one (see eq. (20)):

$$I = \int_0^\infty dx x^2 K_1(x) \int_0^\infty dy y^2 K_1(y) \int_0^\infty dz z^2 K_1(z) \times \frac{\theta([z^2 - (x-y)^2][(x+y)^2 - z^2])}{\sqrt{[z^2 - (x-y)^2][(x+y)^2 - z^2]}}. \quad (\text{B.1})$$

First we integrate in variable z , using following tabulated integral (eq. 2.16.4.1. in [16], $|a| < b$):

$$\int_a^b dz z K_0(cz) \frac{1}{\sqrt{[z^2 - a^2][b^2 - z^2]}} = \frac{\pi}{2} I_0\left[\frac{(b-a)c}{2}\right] K_0\left[\frac{(b+a)c}{2}\right], \quad (\text{B.2})$$

where $I_\nu(z)$ is the modified Bessel Function of the first kind. Let us differentiate both sides of eq. (B.2) in parameter c and then put $c = 1$, $a = x - y$, $b = x + y$. As a result, we get

$$\begin{aligned} & \int_0^\infty dz z^2 K_1(z) \frac{\theta([z^2 - (x-y)^2][(x+y)^2 - z^2])}{\sqrt{[z^2 - (x-y)^2][(x+y)^2 - z^2]}} \\ &= \frac{\pi}{2} [x K_1(x) I_0(y) - K_0(x) y I_1(y)]. \end{aligned} \quad (\text{B.3})$$

It was taken into account that $K'_0(z) = -K_1(z)$, $I'_0(z) = I_1(z)$. Thus, our integral looks like

$$\begin{aligned} I = \frac{\pi}{2} & \left[\int_0^\infty dx x^3 [K_1(x)]^2 \int_0^x dy y^2 K_1(y) I_0(y) \right. \\ & \left. - \int_0^\infty dx x^2 K_1(x) K_0(x) \int_0^x dy y^3 K_1(y) I_1(y) \right] = \frac{\pi}{2} (I_1 - I_2), \end{aligned} \quad (\text{B.4})$$

where

$$I_1 = \int_0^\infty dx x^3 [K_1(x)]^2 \int_0^x dy y^2 K_1(y) I_0(y), \quad (\text{B.5})$$

$$I_2 = \int_0^\infty dx x^3 K_1(x) I_1(x) \int_x^\infty dy y^2 K_1(y) K_0(y). \quad (\text{B.6})$$

The following integrals are known (see eqs. 1.12.4.2 and 1.12.3.1 in [16]):

$$\int_0^x dy y I_0(ay) K_0(by) = \frac{1}{b^2 - a^2} - \frac{x}{b^2 - a^2} [a I_1(ax) K_0(bx) + b I_0(ax) K_1(bx)] , \quad (\text{B.7})$$

$$\int_x^\infty dy y K_0(ay) K_0(by) = \frac{x}{a^2 - b^2} [a K_1(ax) K_0(bx) - b K_0(ax) K_1(bx)] , \quad (\text{B.8})$$

Let us differentiate both sides of these equations in parameter b and after that put $a = b = 1$. As a result, we find

$$\int_0^x dy y^2 K_1(y) I_0(y) = \frac{x^2}{4} [2K_1(x) I_1(x) + 1] , \quad (\text{B.9})$$

$$\int_x^\infty dy y^2 K_1(y) K_0(y) = \frac{x^2}{2} [K_1(x)]^2 . \quad (\text{B.10})$$

Since [16]

$$I_1 - I_2 = \frac{1}{4} \int_0^\infty dx x^5 [K_1(x)]^2 = \frac{2}{5} , \quad (\text{B.11})$$

we obtain $I = \pi/5$.

The second integral we are interested in (see eq. (48)) is the following:

$$J = \int_0^\infty dx x e^{-x} \int_0^\infty dy y e^{-y} \int_0^\infty dz z e^{-z} \times \frac{\theta([z^2 - (x - y)^2][(x + y)^2 - z^2])}{\sqrt{[z^2 - (x - y)^2][(x + y)^2 - z^2]}} . \quad (\text{B.12})$$

Let us define new variables

$$u = x + y , \quad v = x - y . \quad (\text{B.13})$$

Integrating in variable v first and then using substitution $z = ut$, we obtain:

$$\begin{aligned} J &= \frac{1}{4} \int_0^\infty du e^{-u} \int_z^u dz z e^{-z} \int_0^z dv \left[\sqrt{\frac{u^2 - z^2}{z^2 - v^2}} + \sqrt{\frac{z^2 - v^2}{u^2 - z^2}} \right] \\ &= \frac{\pi}{16} \int_0^1 dt t \frac{2 - t^2}{\sqrt{1 - t^2}} \int_0^\infty du u^3 e^{-u(1+t)} = \frac{3\pi}{8} \int_0^1 dt \frac{t(2 - t^2)}{(1 + t)^4 \sqrt{1 - t^2}} \\ &= \frac{3\pi}{8} \times \frac{4}{21} = \frac{\pi}{14} . \end{aligned} \quad (\text{B.14})$$

The values of the integrals I and J are needed to evaluate the term σ_3 in expansion (6).

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